Kvantna dinamika i slike Verovatnoće u "kasnijem trenutku"

1) Šredingerova j-na

$$i\hbar \frac{d|\psi(t)\rangle}{dt} = \hat{H}|\psi(t)\rangle$$

ili $$|\psi(t)\rangle = \hat{U}\,|\psi(t_0)\rangle$$

$\hat{U}$ → unitarni operator vremenske evolucije

2) Za konzervativne sisteme $\hat{H} \neq f(t)$

$$\hat{U}(t-t_0, t_0) = e^{-\frac{i}{\hbar}(t-t_0)\hat{H}}$$

$$\hat{H}|\psi_n\rangle = E_n|\psi_n\rangle \quad \text{(stacionarna)}$$

3) Slike
- Šredingerova $|\psi(t)\rangle$, $\hat{A}_S$
- Hajzenbergova $\hat{A}_H(t)$, $|\psi\rangle$

Ove dve slike su fizički ekvivalentne

4) Verovatnoća dobijanja diskretne vrednosti a_n opservable $\hat{A}$ u kasnijem trenutku t, data je izrazom

$$W(\hat{A}, a_n, |\psi\rangle, t) = \langle\psi(t)|\hat{P}_n|\psi(t)\rangle$$

1. Dokazati unitarnost operatora vremenske evolucije koji je za konzervativni sistem dat izrazom:

$$\hat{U}(t) = e^{-\frac{i}{\hbar}\hat{H}t}$$

gde je $\hat{H}$ - Hamiltonijan sistema.

$$\hat{U}(t)\hat{U}^{+}(t) = \hat{U}^{+}(t)\hat{U}(t) = \hat{I}$$

$$\hat{U}^{+}(t) = e^{\frac{i}{\hbar}\hat{H}t}$$

$$\hat{U}(t)\hat{U}^{+}(t) = e^{-\frac{i}{\hbar}\hat{H}t} e^{\frac{i}{\hbar}\hat{H}t} = e^{-\frac{i}{\hbar}\hat{H}t + \frac{i}{\hbar}\hat{H}t + \frac{1}{2}[-\frac{i}{\hbar}\hat{H}t, \frac{i}{\hbar}\hat{H}t}$$

$$= \hat{I}$$

$\hat{U}^{+}(t)\hat{U}(t) = \hat{I}$ $\qquad$ $\hat{U}(t)$ jeste unitaran operator

$\hat{U}(t)\hat{U}^{+}(t') \neq \hat{I}$ $\qquad$ $t \neq t'$

2. Razviti $\hat{U}$ definisan u prethodnom zadatku u red po malom parametru t i proveriti unitarnost.

$$\hat{U}(t) = e^{-\frac{i}{\hbar}\hat{H}t}$$

$$\hat{U}(t) = \sum_{n=0}^{\infty} \frac{\left(-\frac{i}{\hbar}t\right)^n \hat{H}^n}{n!} = \hat{I} - \frac{1}{1!}\frac{i}{\hbar}t\hat{H} + \frac{1}{2!}\left(-\frac{it}{\hbar}\right)^2\hat{H}^2 + \dots$$

$$t \ll 1 \Rightarrow t^2 \to 0$$

$$= \hat{I} - \frac{it}{\hbar}\hat{H} + O(t^2) \simeq \hat{I} - \frac{it}{\hbar}\hat{H}$$

$$\hat{U}^{+}(t) = \hat{I} + \frac{it}{\hbar}\hat{H}$$

Da li je $\hat{U}\hat{U}^{+} = \hat{I}$?

$$\hat{U}\hat{U}^{+} = \left(\hat{I} - \frac{it}{\hbar}\hat{H}\right)\left(\hat{I} + \frac{it}{\hbar}\hat{H}\right) = \hat{I} + \frac{t^2}{\hbar^2}\hat{H}^2 = \hat{I}$$

a lako se vidi da važi i

$$\hat{U}^{+}\hat{U} = \hat{I}$$

3. Rešiti stacionarnu ŠJ za trodimenzionalnu slobodnu česticu.

Već je rešavano za 1D

$$\hat{H} = \frac{\hat{p}_x^2}{2m}, \qquad \hat{H}|\Psi\rangle = E|\Psi\rangle$$

$$[\hat{H}, \hat{p}_x] = 0$$

$$\hat{p}_x |p_x\rangle = p_x |p_x\rangle \longrightarrow \Psi_{p_x}(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar} x p_x}$$

$$\left\{ \begin{array}{l} \hat{p}_x^2 |p_x\rangle = p_x^2 |p_x\rangle \\ \hat{H}|\Psi\rangle = E|\Psi\rangle \end{array} \right.$$ u koordinatnoj reprezentaciji

$$-\hbar^2 \frac{d^2}{dx^2} \Psi_{p_x}(x) = p_x^2 \Psi_{p_x}(x) \qquad (*)$$

Rešenje ove jednačine

$$\Psi_{p_x}(x) = c_1 e^{-\frac{i}{\hbar} x p_x} + c_2 e^{\frac{i}{\hbar} x p_x}$$, ali zato

što je $[\hat{H}, \hat{p}_x] = 0$ $\Psi_{p_x}(x)$ za $(*)$ je $\frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar} x p_x}$

Dakle,

sv. vrednost $E_x = \frac{p_x^2}{2m}$

svoj. stanje $\Psi_{p_x}(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar} x p_x}$

Isto ~~za~~ i za faktor prostore y i z.

$$\hat{H} = \frac{\hat{\vec{P}}^2}{2m} = \frac{1}{2m}\left(\hat{P}_x^2 \otimes \hat{I}_y \otimes \hat{I}_z + \hat{I}_x \otimes \hat{P}_y^2 \otimes \hat{I}_z + \hat{I}_x \otimes \hat{I}_y \otimes \hat{P}_z^2\right)$$

$$\hat{H}|\psi\rangle = E|\psi\rangle$$

$$|\psi\rangle = |\psi_{px}\rangle|\psi_{py}\rangle|\psi_{pz}\rangle$$

$$E = \frac{\vec{P}^2}{2m}, \quad \psi(\vec{r}) = \psi_{px}(x)\,\psi_{py}(y)\,\psi_{pz}(z)$$

$$= \frac{1}{(2\pi\hbar)^{\frac{3}{2}}} e^{\frac{i}{\hbar}\vec{r}\cdot\vec{p}}$$

[Napomena: Sv. problem za rotator u ravni i prostoru se radi u $\underline{\overline{IV}}$ glavi]

4. Koristeći se ŠJ, eksplicitnim računom dokazati važenje jednačine kontinuiteta za slobodnu česticu (3D). Da li prisustvo spoljašnjeg potencijala utiče na oblik struje?

$$i\hbar \frac{\partial |\psi(t)\rangle}{\partial t} = \hat{H} |\psi(t)\rangle$$

1D slučaj

$$\hat{H} = \frac{\hat{p}_x^2}{2m} + V(\hat{x},t) \quad \rightarrow \text{tipični Hamiltonijan}$$

Koordinatna reprezentacija

$$i\hbar \frac{\partial \langle x|\psi(t)\rangle}{\partial t} = \langle x|\hat{H}|\psi(t)\rangle$$

$$= \langle x|\hat{H} \int_{-\infty}^{+\infty} |x'\rangle\langle x'| dx' |\psi(t)\rangle$$

$$= \int_{-\infty}^{+\infty} \langle x|\hat{H}|x'\rangle \langle x'|\psi(t)\rangle dx'$$

$$\langle x|\hat{H}|x'\rangle = \frac{1}{2m} \langle x|\hat{p}_x^2|x'\rangle + \langle x|V(\hat{x},t)|x'\rangle$$

$$= \frac{1}{2m}(-\hbar^2)\overset{\delta(x-x')}{} \frac{\partial^2}{\partial x'^2} + \delta(x-x')V(x',t)$$

$$= \int_{-\infty}^{+\infty} \left(-\frac{\hbar^2}{2m}\delta(x-x')\frac{\partial^2}{\partial x'^2} + \delta(x-x')V(x',t)\right)\psi(x',t)dx'$$

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi(x,t)}{\partial x^2} + V(x,t)\psi(x,t) \qquad (1)$$

$$W(\hat{X}, x \in (\alpha, \beta], |\psi(t)\rangle) = \langle \psi(t)| \int_\alpha^\beta |x\rangle\langle x| \, dx \, |\psi(t)\rangle$$

$$= \int_\alpha^\beta \underbrace{|\psi(x,t)|^2}_{\text{gustina verovatnoće}} dx$$

$$\rho(x,t) = |\psi(x,t)|^2 \quad \text{odnosno} \quad \rho(x,t) = \psi^*(x,t)\,\psi(x,t)$$

$$\frac{\partial \rho(x,t)}{\partial t} = \frac{\partial \psi^*(x,t)}{\partial t}\psi(x,t) + \psi^*(x,t)\frac{\partial \psi(x,t)}{\partial t} \quad (2)$$

$$(1)|^* \Rightarrow$$

$$-i\hbar \frac{\partial \psi^*(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi^*(x,t)}{\partial x^2} + V(x,t)\psi^*(x,t) \Rightarrow$$

$$i\hbar \frac{\partial \psi^*(x,t)}{\partial t} = \frac{\hbar^2}{2m}\frac{\partial^2 \psi^*(x,t)}{\partial x^2} - V(x,t)\psi^*(x,t) \quad (3)$$

$$(1) \wedge (3) \cup (2) \Rightarrow \text{(pomnožemo sa } i\hbar)$$

$$i\hbar \frac{\partial \rho(x,t)}{\partial t} = \left(\frac{\hbar^2}{2m}\frac{\partial^2 \psi^*(x,t)}{\partial x^2} - V(x,t)\psi^*(x,t)\right)\psi(x,t) +$$

$$+ \psi^*(x,t)\left(-\frac{\hbar^2}{2m}\frac{\partial^2 \psi(x,t)}{\partial x^2} + V(x,t)\psi(x,t)\right)$$

$$= \frac{\hbar^2}{2m}\left(\frac{\partial^2 \psi^*(x,t)}{\partial x^2}\psi(x,t) - \psi^*(x,t)\frac{\partial^2 \psi(x,t)}{\partial x^2}\right)$$

$$= \frac{\hbar^2}{2m}\frac{\partial}{\partial x}\left(\frac{\partial \psi^*(x,t)}{\partial x}\psi(x,t) - \psi^*(x,t)\frac{\partial \psi(x,t)}{\partial x}\right)$$

$$\frac{\partial \rho(x,t)}{\partial t} = \frac{\hbar}{2mi}\frac{\partial}{\partial x}\left[\frac{\partial \Psi^*(x,t)}{\partial x}\Psi(x,t) - \Psi^*(x,t)\frac{\partial \Psi(x,t)}{\partial x}\right]$$

$$\frac{\partial \rho(x,t)}{\partial t} = -\frac{\hbar}{2mi}\frac{\partial}{\partial x}\left[\Psi^*(x,t)\frac{\partial \Psi(x,t)}{\partial x} - \Psi(x,t)\frac{\partial \Psi^*(x,t)}{\partial x}\right]$$

Uopštenje na 3D

$$\frac{\partial \rho(\vec{r},t)}{\partial t} = -\frac{\hbar}{2mi}\vec{\nabla}\left[\Psi^*(\vec{r},t)\vec{\nabla}\Psi(\vec{r},t) - \Psi(\vec{r},t)\vec{\nabla}\Psi^*(\vec{r},t)\right]$$

$$\frac{\partial \rho(\vec{r},t)}{\partial t} + \frac{\hbar}{2mi}\vec{\nabla}\left[\Psi^*(\vec{r},t)\vec{\nabla}\Psi(\vec{r},t) - \Psi(\vec{r},t)\vec{\nabla}\Psi^*(\vec{r},t)\right] = 0$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0 \qquad \text{j-na kontinuiteta}$$

$$\underbrace{\vec{j} = \frac{\hbar}{2mi}\left(\Psi^*(\vec{r},t)\vec{\nabla}\Psi(\vec{r},t) - \Psi(\vec{r},t)\vec{\nabla}\Psi^*(\vec{r},t)\right)}_{\text{gustina struje}}$$

Za slobodnu česticu partikularno rešenje

$$\Psi(\vec{r},t) = \frac{1}{(2\pi\hbar)^{3/2}}e^{\frac{i}{\hbar}\vec{r}\cdot\vec{p}}$$

$$\Psi^*(\vec{r},t) = \frac{1}{(2\pi\hbar)^{3/2}}e^{-\frac{i}{\hbar}\vec{r}\cdot\vec{p}}$$

$$\vec{\nabla}\psi(\vec{r},t) = \frac{i}{\hbar}\vec{p}\,\frac{1}{(2\pi\hbar)^{3/2}}\,e^{\frac{i}{\hbar}\vec{r}\cdot\vec{p}}$$

$$\vec{\nabla}\psi^*(\vec{r},t) = -\frac{i}{\hbar}\vec{p}\,\frac{1}{(2\pi\hbar)^{3/2}}\,e^{-\frac{i}{\hbar}\vec{r}\cdot\vec{p}}$$

$$\vec{j} = \frac{\hbar}{2mi}\Big(\frac{1}{(2\pi\hbar)^{3/2}}\,e^{-\frac{i}{\hbar}\vec{r}\cdot\vec{p}}\;\frac{i}{\hbar}\vec{p}\,\frac{1}{(2\pi\hbar)^{3/2}}\,e^{\frac{i}{\hbar}\vec{r}\cdot\vec{p}} +$$

$$\frac{1}{(2\pi\hbar)^{3/2}}\,e^{\frac{i}{\hbar}\vec{r}\cdot\vec{p}}\;\frac{i}{\hbar}\vec{p}\,\frac{1}{(2\pi\hbar)^{3/2}}\,e^{-\frac{i}{\hbar}\vec{r}\cdot\vec{p}}\Big)$$

$$\vec{j} = \frac{\cancel{\hbar}}{2m\cancel{i}}\,2\,\frac{\cancel{i}}{\cancel{\hbar}}\vec{p}\,\frac{1}{(2\pi\hbar)^3}$$

$$= \frac{1}{(2\pi\hbar)^3}\,\frac{\vec{p}}{m}$$

$$\rho = \psi^*(\vec{r},t)\,\psi(\vec{r},t) = \frac{1}{(2\pi\hbar)^3}$$

$$\vec{j} = \rho\,\frac{\vec{p}}{m} = \rho\vec{v}$$

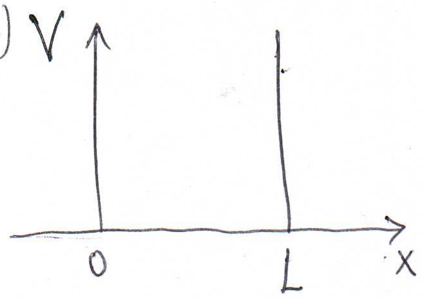
Veza između $\vec{j}$ i ρ je kao u EM za naelektrisanu česticu koja se kreće brzinom $\vec{v}$.

5. Rešiti stacionarnu ŠJ za česticu zarobljenu u beskonačno dubokoj potencijalnoj jami čije je dno sa energijom nula a ivice

a) $[0, L]$ $\quad V(x) = \begin{cases} 0 & 0 < x < L \\ \infty & \text{van} \end{cases}$

b) $[-L, L]$ $\quad V(x) = \begin{cases} 0 & -L < x < L \\ \infty & \text{van} \end{cases}$

a)



$$\left.\begin{aligned} \hat{H}|\psi\rangle &= E|\psi\rangle \\ \hat{H} &= \frac{\hat{p}_x^2}{2m} + V(\hat{x}) \end{aligned}\right\} \overset{\text{koor. repr.}}{\Longrightarrow} -\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2} + V(x)\psi(x) = E\psi(x) \quad (*)$$

$$(*) \Rightarrow \quad -\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2} + (V-E)\psi(x) = 0$$

$$\boxed{V = 0 \qquad x \in [0, L]}$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2} - E\psi(x) = 0$$

$$\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2} + E\psi(x) = 0$$

$$\psi'' + \frac{2mE}{\hbar^2}\psi = 0 \quad ; \quad \alpha^2 = \frac{2mE}{\hbar^2}$$

$$\psi'' + \alpha^2\psi = 0$$

$$k^2 + \alpha^2 = 0 \quad \Rightarrow \quad k_{1,2} = \pm i\alpha$$

$$\psi(x) = c_1' e^{i\alpha x} + c_2' e^{-i\alpha x}$$

$$\boxed{\psi(x) = c_1 \cos \alpha x + c_2 \sin \alpha x}$$ Opšte rešenje

$$\psi(0) = 0 \Rightarrow 0 = c_1$$

$$\psi(L) = 0 \Rightarrow \underline{0} = c_1 \cos \alpha L + c_2 \sin \alpha L = \underline{c_2 \sin \alpha L}$$

Dakle

$$\psi(x) = c_2 \sin \alpha x$$

$$c_2 \sin \alpha L = 0 \Rightarrow \alpha L = n\pi \quad (n = 0, 1, 2, \dots) \Rightarrow$$

$$\boxed{\alpha = \frac{\pi n}{L}} \quad (**)$$

$$\psi(x) = c_2 \sin\left(\frac{\pi n}{L} x\right)$$

$$\alpha^2 = \frac{2mE}{\hbar^2} \Rightarrow E = \frac{\hbar^2 \alpha^2}{2m} = \frac{\hbar^2 \pi^2 n^2}{2m L^2} \equiv E_n$$

Kvantovana energija

Normiranje

$$\int_0^L |\psi(x)|^2 dx = 1$$

$$|c_2|^2 \int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx = 1$$

$$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$$

$$\int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx = \frac{1}{2}\int_0^L \left(1 - \cos\frac{2n\pi x}{L}\right) dx$$

$$= \frac{1}{2}\left(L - \int_0^L \cos\frac{2n\pi x}{L}\, dx\right)$$

$$= \frac{1}{2}\left(L - \frac{L}{2n\pi}\int_0^L \cos\frac{2n\pi x}{L}\, d\left(\frac{2n\pi x}{L}\right)\right)$$

$$= \frac{1}{2}\left(L - \frac{L}{2n\pi}\sin\frac{2n\pi x}{L}\Big|_0^L\right)$$

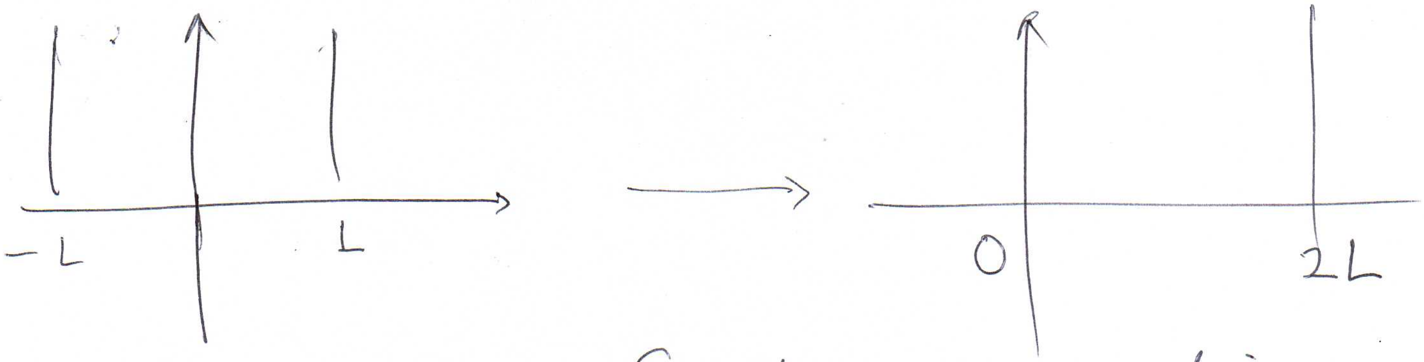
$$= \frac{1}{2}\left(L - \frac{L}{2n\pi}(0-0)\right) = \frac{L}{2}$$

$$|c_2|^2 \frac{L}{2} = 1 \Rightarrow |c_2|^2 = \frac{2}{L} \Rightarrow c_2 = \sqrt{\frac{2}{L}}$$

Konačno,

$$\boxed{\begin{aligned} \psi(x) &= \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi}{L}x\right) \\ E_n &= \frac{\hbar^2\pi^2 n^2}{2mL^2} \end{aligned}}$$

b) Razmatramo rešenje iz a)



ali translirano. Svojstvene vrednosti ne smeju da zavise od izbora koordinatnog sistema.

$$L \to 2L$$

$$x \to x + L$$

$$\psi(x) = \sqrt{\frac{1}{L}} \sin\left(\frac{n\pi}{2L}(x+L)\right)$$

$$= \sqrt{\frac{1}{L}} \sin\left(\frac{n\pi x}{2L} + \frac{n\pi}{2}\right)$$

$n = 2k$

$n = 2k+1$

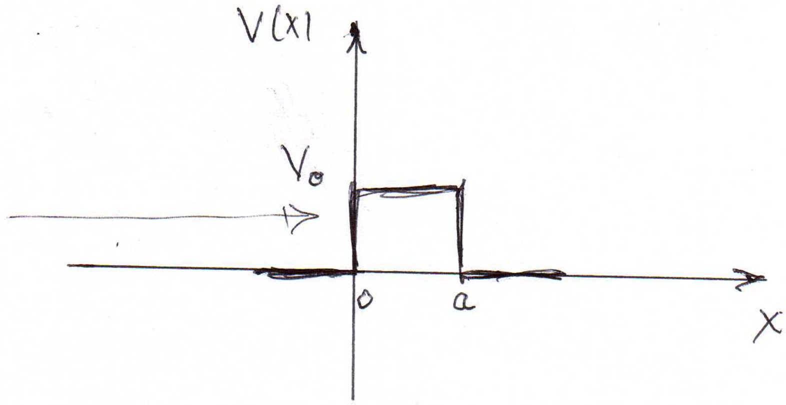
$$\psi(x) = \sqrt{\frac{1}{L}} \begin{cases} \sin\left(\frac{k\pi x}{L}\right) \\ \cos \frac{(2k+1)\pi x}{2L} \end{cases}$$

$$E_n = \frac{\hbar^2 \pi^2 n^2}{8 m L^2}$$

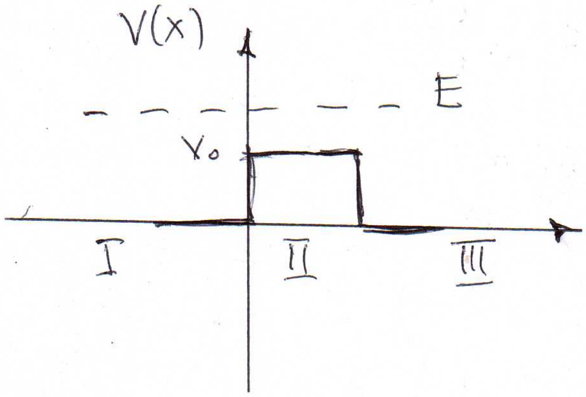
6. Za potencijalnu barijeru datu na slici, proračunati koeficijent prozračnosti barijere. Potencijal je dat izrazom:

$$V(x) = \begin{cases} 0 & x < 0 \\ V_0 & 0 \leq x \leq a \\ 0 & x > a \end{cases}$$

За КТФ само под А?



A) $E > V_0$



$$\psi(x) = \begin{cases} \psi_1(x) = A e^{ik_1 x} + B e^{-ik_1 x} \\ \psi_2(x) = C e^{ik_2 x} + D e^{-ik_2 x} \\ \psi_3(x) = E e^{ik_1 x} \end{cases}$$

$$k_1 = \sqrt{\frac{2mE}{\hbar^2}} \qquad k_2 = \sqrt{\frac{2m(E-V_0)}{\hbar^2}}$$

$\psi(x)$ i $\frac{d\psi(x)}{dx}$ moraju biti neprekidni u tačkama $x=0$ i $x=a$.

1) $\psi_1(0) = \psi_2(0)$ 3) $\frac{d\psi_1(0)}{dx} = \frac{d\psi_2(0)}{dx}$

2) $\psi_2(a) = \psi_3(a)$ 4) $\frac{d\psi_2(a)}{dx} = \frac{d\psi_3(a)}{dx}$

Iz ovih uslova sledi sistem j-na

(1) $A+B = C+D$

(2) $Ce^{ik_2a} + De^{-ik_2a} = Ee^{ik_1a}$

(3) $k_1(A-B) = k_2(C-D)$

(4) $k_2(Ce^{ik_2a} - De^{-ik_2a}) = k_1Ee^{ik_1a}$

Koeficijenti refleksije i propustljivosti

$$R = \left|\frac{\vec{j}_R}{\vec{j}_U}\right| \qquad T = \left|\frac{\vec{j}_T}{\vec{j}_U}\right|$$

U prethodnom zadatku je razmatrana gustina struje. Za 1D slučaj će biti

$$j_{xU} = \frac{\hbar}{2mi}\left(\psi_U^*(x)\frac{\partial\psi_U(x)}{\partial x} - \psi_U(x)\frac{\partial\psi_U^*(x)}{\partial x}\right)$$

$$\psi_U(x) = Ae^{ik_1x} \Rightarrow \psi_U^*(x) = A^*e^{-ik_1x}$$

"U" - upadno

$$j_{xU} = \frac{\hbar}{2mi}\left(A^* e^{-ik_1x}(ik_1)Ae^{ik_1x} - Ae^{ik_1x}A^*(-ik_1)e^{-ik_1x}\right)$$

$$= \frac{\hbar}{2mi}\left(|A|^2 ik_1 + |A|^2 ik_1\right)$$

$$= \frac{\hbar}{\cancel{2}m\cancel{i}}\, \cancel{2}|A|^2 \cancel{i} k_1$$

$$= \frac{\hbar k_1}{m} |A|^2$$

$\psi_R(x) = Be^{-ik_1x}$ a j_{xR} se računa kao i prethodno i glasi:

$j_{xR} = -\frac{\hbar k_1}{m}|B|^2$, a slično i

$j_{xT} = \frac{\hbar k_1}{m}|E|^2$.

Koeficijent prozračnosti će onda biti

$$T = \left|\frac{|E|^2}{|A|^2}\right| = \frac{|E|^2}{|A|^2}$$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, \quad z_1 \in C, \; z_2 \in R$$

$$|z_1 \cdot z_2| = |z_1| \cdot |z_2|$$

Iz sistema jednačina treba naći odnos

$$\frac{E}{A}$$

$$(1) \qquad A + B = C + D$$

$$(3) \qquad k_1 A - k_1 B = k_2 C - k_2 D$$

$$(1)\Big|^{k_2} + (3) \Rightarrow$$

$$2k_2 C = A(k_1 + k_2) + B(k_2 - k_1)$$

$$\boxed{C = \frac{1}{2k_2}\left[A(k_1+k_2) + B(k_2 - k_1)\right]} \qquad (5)$$

$$(1)\Big|^{(-k_2)} + (3) \Rightarrow$$

$$-2k_2 D = A(k_1 - k_2) - B(k_2 + k_1)$$

$$\boxed{D = \frac{1}{2k_2}\left[B(k_1+k_2) + A(k_2 - k_1)\right]} \qquad (6)$$

(5) i (6) zamenti v (2)

$$\frac{1}{2k_2}\left[A(k_1+k_2) + B(k_2-k_1)\right]e^{ik_2 a} + \frac{1}{2k_2}\left[B(k_1+k_2) + A(k_2-k_1)\right]e^{-ik_2 a} =$$

$$= F e^{ik_1 a}$$

a odozgo

$$2k_2 E e^{ik_1 a} = A\left[(k_1+k_2)e^{ik_2 a} + (k_2-k_1)e^{-ik_2 a}\right] +$$

$$+ B\left[(k_2-k_1)e^{ik_2 a} + \cancel{B}(k_2+k_1)e^{-ik_2 a}\right] \qquad (7)$$

Kada se (5) i (6) zameni u (4) dobija se

$$2k_1 E e^{ik_1 a} = A\left[(k_1+k_2)e^{ik_2 a} - (k_2-k_1)e^{-ik_2 a}\right] +$$

$$B\left[(k_2-k_1)e^{ik_2 a} - (k_1+k_2)e^{-ik_2 a}\right] \quad (8)$$

Zapišimo (7) i (8) u koncitnijem obliku

$$\alpha_1 E = a_1 A + b_1 B \qquad (7')$$

$$\alpha_2 E = a_2 A + b_2 B \qquad (8')$$

$$(7') \,\Big|\, \left(-\frac{b_2}{b_1}\right) + (8') \implies$$

$$\frac{E}{A} = \frac{b_1 a_2 - b_2 a_1}{\alpha_2 b_1 - \alpha_1 b_2}$$

$$\alpha_1 = 2k_2 e^{ik_1 a}$$

$$a_1 = (k_1+k_2)e^{ik_2 a} + (k_2-k_1)e^{-ik_2 a}$$

$$b_1 = (k_2-k_1)e^{ik_2 a} + (k_1+k_2)e^{-ik_2 a}$$

$$\alpha_2 = 2k_1 e^{ik_1 a}$$

$$a_2 = (k_1+k_2)e^{ik_2 a} - (k_2-k_1)e^{-ik_2 a}$$

$$b_2 = (k_2-k_1)e^{ik_2 a} - (k_2+k_1)e^{-ik_2 a}$$

$$\frac{E}{A} = \frac{\left[(k_2-k_1)\,e^{ik_2a} + (k_1+k_2)\,e^{-ik_2a}\right]\left[(k_1+k_2)\,e^{ik_2a} - (k_2-k_1)\,e^{-ik_2a}\right] - \left[(k_2-k_1)\,e^{ik_2a} - (k_2+k_1)\,e^{-ik_2a}\right]\left[(k_1+k_2)e^{ik_2a} + (k_2-k_1)e^{-ik_2}\right]}{2k_1\,e^{ik_1a}\left[(k_2-k_1)e^{ik_2a} + (k_1+k_2)e^{-ik_2a}\right] - 2k_2\,e^{ik_1a}\left[(k_2-k_1)\,e^{ik_2a} - (k_2+k_1)\,e^{-ik_2a}\right]}$$

$$= \frac{\cancel{(k_2^2-k_1^2)\,e^{2ik_2a}} - (k_2-k_1)^2 + (k_1+k_2)^2 - \cancel{(k_2^2-k_1^2)\,e^{-2ik_2a}} - \cancel{(k_2^2-k_1^2)\,e^{2ik_2a}} - (k_2-k_1)^2 + (k_2+k_1)^2 + \cancel{(k_2^2-k_1^2)\,e^{-2ik_2}}}{2\,e^{ik_1a}\left(k_1(k_2-k_1)\,e^{ik_2a} - k_2(k_2-k_1)\,e^{ik_2a} + k_1(k_1+k_2)\,e^{-ik_2a} + k_2(k_2+k_1)\,e^{-ik_2a}\right)}$$

$$= \frac{\cancel{2}\left[(k_1+k_2)^2 - (k_2-k_1)^2\right]}{\cancel{2}\;e^{ik_1a}\left(\left(k_1k_2 - k_1^2 - k_2^2 + k_2k_1\right)e^{ik_2a} + (k_1+k_2)(k_1+k_2)\,e^{-ik_2a}\right)}$$

$$= \frac{(k_1+k_2)^2 - (k_2-k_1)^2}{e^{ik_1a}\left(-(k_2-k_1)^2\,e^{ik_2a} + (k_1+k_2)^2\,e^{-ik_2a}\right)}$$

$$= e^{-ik_1a}(4k_1k_2)\left[(k_1+k_2)^2e^{-ik_2a} - (k_2-k_1)^2\,e^{ik_2a}\right]^{-1} \qquad (9)$$

$$|E|^2 = |A|^2\,|4k_1k_2|^2\,\left|\left[(k_1+k_2)^2\,e^{-ik_2a} - (k_2-k_1)^2\,e^{ik_2a}\right]^{-1}\right|^2$$

odnosno upotrebom Euler-ovog obrasca

$$\frac{|E|^2}{|A|^2} = (4k_1k_2)^2 \frac{1}{16k_1^2k_2^2\cos^2(k_2a) + 4(k_1^2+k_2^2)^2\sin^2(k_2a)}$$

odnosno

$$\frac{|E|^2}{|A|^2} = (4k_1k_2)^2 \frac{1}{16k_1^2k_2^2 + 4(k_1^2-k_2^2)^2\sin^2(k_2a)}$$

i konačno

$$\left|\frac{|E|^2}{|A|^2}\right| = \left|\frac{1}{1 + \frac{1}{4}\left(\frac{k_1^2-k_2^2}{k_1k_2}\right)^2\sin^2 k_2a}\right| = T$$

Za domaći: Izračunati R

Kreće se od (7') i (8')

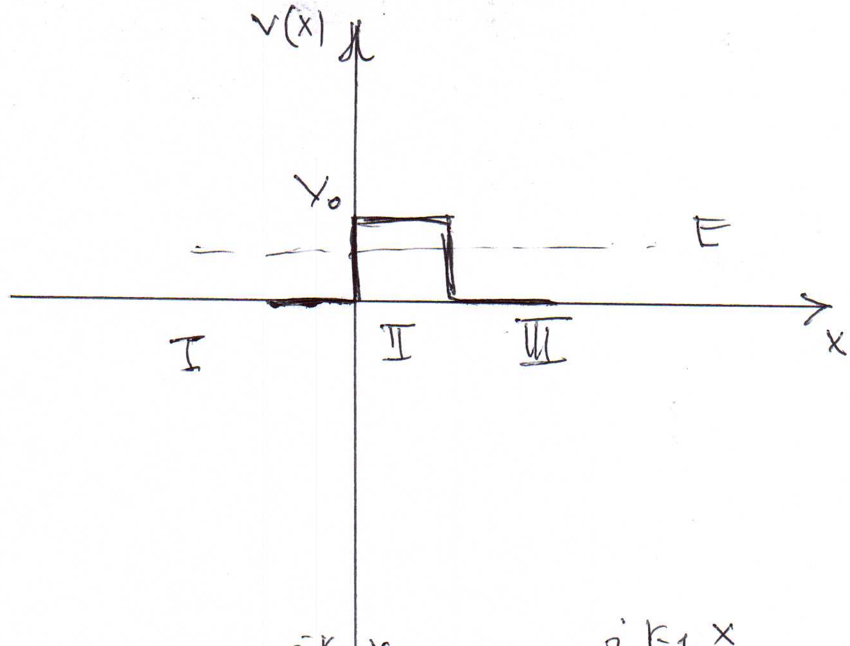
$$(7)' / \left(-\frac{\alpha_2}{\alpha_1}\right) + (8') \Rightarrow$$

$$B = A\,\frac{a_2\alpha_1 - \alpha_2 a_1}{\alpha_2 b_1 - b_2\alpha_1}$$

Dobijeni rezultat bi trebalo proveriti uz pomoć relacije $T+R=1$

Komentar: T u KM može biti manji od jedinice!

B) $E < V_0$



$$\Psi(x) = \begin{cases} \Psi_1(x) = A\, e^{ik_1 x} + B\, e^{-ik_1 x} \\ \Psi_2(x) = C\, e^{k_2 x} + D\, e^{-k_2 x} \\ \Psi_3(x) = E\, e^{ik_1 x} \end{cases}$$

$$k_1 = \sqrt{\frac{2mE}{\hbar^2}} \qquad k_2 = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}$$

$$T = \frac{1}{1 + \frac{1}{4}\left(\frac{k_1^2 - k_2^2}{k_1 k_2}\right)^2 \sin^2 k_2 a}$$ za prethodni slučaj

Ovo k_2 i ono iz A) su isti do na kompleksnu jedinicu $k_2 \to i k_2$ $(\operatorname{sh} x = -i \sin(ix))$

$$T = \frac{1}{1 + \frac{1}{4}\left(\frac{k_1^2 + k_2^2}{k_1 k_2 i}\right)^2 \sin^2 (i k_2 a)}$$

$$T = \frac{1}{1 + \frac{1}{4}\left(\frac{k_1^2 + k_2^2}{k_1 k_2}\right)^2 sh^2(k_2 a)}$$

Dakle, $T \neq 0 \iff$ kvantno-mehanički efekat

$\boxed{\text{TUNELOVANJE}}$

Domaći: Koliko iznosi R?

7. Koje uslove mora da zadovoljava opservabla $\hat{A}$, da bi bila "integral kretanja" datog sistema?

$$\hat{A}_H(t) = \hat{U}^+(t)\,\hat{A}_S(t_0)\,\hat{U}(t) \quad ; \quad \hat{A}_H(t_0) = \hat{A}_S(t_0) = \hat{A}_S(t) \equiv \hat{A}_S$$

$$i\hbar \frac{d\hat{A}_H(t)}{dt} = [\hat{A}_H, \hat{H}_H] + i\hbar \hat{U}^+(t) \frac{\partial \hat{A}_S}{\partial t} \hat{U}(t)$$

Ako je $\frac{\partial \hat{A}_S}{\partial t} = 0 \Rightarrow$

$$i\hbar \frac{d\hat{A}_H(t)}{dt} = [\hat{A}_H, \hat{H}_H] \qquad \hat{A}_H \equiv \hat{A}_H(t), \quad \hat{H}_H \equiv \hat{H}_H(t)$$

Integral kretanja

$$\frac{d\hat{A}_H(t)}{dt} = 0 \Rightarrow$$

$$[\hat{A}_H, \hat{H}_H] = 0$$

$$\Downarrow$$

$$[\hat{U}^+\hat{A}_S\hat{U}, \hat{U}^+\hat{H}_S\hat{U}] = \hat{U}^+\hat{A}_S\hat{U}\hat{U}^+\hat{H}_S\hat{U} - \hat{U}^+\hat{H}_S\hat{U}\hat{U}^+\hat{A}_S\hat{U}$$

$$= \hat{U}^+\hat{A}_S\hat{H}_S\hat{U} - \hat{U}^+\hat{H}_S\hat{A}_S\hat{U} = \hat{U}^+[\hat{A}_S, \hat{H}_S]\hat{U} = 0 \Rightarrow$$

$$[\hat{A}_S, \hat{H}_S] = 0$$

Opservabla koja komutira sa Hamiltonijanom sistema je "integral kretanja"

9. Izračunati $\hat{U}^+(t)\hat{\vec{r}}\hat{U}(t)$ za

a) slobodnu česticu

b) LHO ~~(ne KTF)~~

a) $\hat{U}(t) = e^{-\frac{i}{\hbar}t\hat{H}_S} = e^{-\frac{i}{\hbar}t\frac{\hat{\vec{p}}_S^{\,2}}{2m}}$; $\hat{\vec{r}}_S = \hat{\vec{r}}$, $\hat{\vec{p}}_S = \hat{\vec{p}}_S$

$$\hat{U}^+(t)\hat{\vec{r}}\hat{U}(t) = e^{\frac{i}{\hbar}t\hat{H}_S}\hat{\vec{r}}e^{-\frac{i}{\hbar}t\hat{H}_S} = \hat{\vec{r}}_H(t)$$

$$\hat{\vec{r}}_H(t) = e^{\frac{i}{\hbar}t\frac{\hat{\vec{p}}_S^{\,2}}{2m}}\hat{\vec{r}}_S e^{-\frac{i}{\hbar}t\frac{\hat{\vec{p}}_S^{\,2}}{2m}} =$$

$$e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + [\hat{A},\hat{B}] + \frac{1}{2!}[\hat{A},[\hat{A},\hat{B}]] + \frac{1}{3!}[\hat{A},[\hat{A},[\hat{A},\hat{B}]]]$$

$$\hat{\vec{r}}_H(t) = \hat{\vec{r}}_S + \frac{i}{\hbar}t\left[\frac{\hat{\vec{p}}_S^{\,2}}{2m},\hat{\vec{r}}_S\right] - \frac{1}{2!}\frac{t^2}{\hbar^2}\left[\frac{\hat{\vec{p}}_S}{2m},\left[\frac{\hat{\vec{p}}_S^{\,2}}{2m},\hat{\vec{r}}_S\right]\right] + \dots$$

$$\left[\frac{\hat{\vec{p}}_S^{\,2}}{2m},\hat{\vec{r}}\right] = -\frac{1}{2m}[\hat{\vec{r}}_S,\hat{\vec{p}}_S^{\,2}] = -\frac{1}{2m}2i\hbar\hat{\vec{p}}_S = -\frac{i\hbar}{m}\hat{\vec{p}}_S$$

a ostali komutatori su nula

$$\hat{\vec{r}}_H(t) = \hat{\vec{r}}_S + \frac{i}{\hbar}t\left(-\frac{i\hbar}{m}\hat{\vec{p}}_S\right) = \hat{\vec{r}}_S + \frac{t}{m}\hat{\vec{p}}_S$$

b) Domaći

$$\hat{x}_H(t) = \hat{x}_S\cos\omega t + \frac{\hat{p}_S}{m\omega}\sin\omega t$$

$$\hat{y}_H(t) = \hat{y}_S$$

$$\hat{z}_H(t) = \hat{z}_S$$

8. Izračunati $\hat{U}^{+}(t)\,\hat{\vec{P}}_S\,\hat{U}(t)$ za

a) slobodnu česticu КТФ (32 вежбе

b) LHO

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a) $\hat{H}_S = \dfrac{\hat{\vec{P}}_S^{\,2}}{2m}$

$$\hat{U} = e^{-\frac{i}{\hbar} t \frac{\hat{\vec{P}}_S^{\,2}}{2m}} \Rightarrow \hat{U}^{+} = e^{\frac{i}{\hbar} t \frac{\hat{\vec{P}}_S^{\,2}}{2m}}$$

$$\hat{\vec{P}}_H(t) = e^{\frac{i}{\hbar} t \frac{\hat{\vec{P}}_S^{\,2}}{2m}}\,\hat{\vec{P}}_S\,e^{-\frac{i}{\hbar} t \frac{\hat{\vec{P}}_S^{\,2}}{2m}}$$

$$\hat{U} = f(\hat{\vec{P}}_S) \Rightarrow [\hat{U}, \hat{\vec{P}}_S] = 0 \quad ; \quad \frac{\partial \hat{\vec{P}}_S}{\partial t} = 0$$

$$\hat{P}_H(t) = \hat{\vec{P}}_S$$

b) $$\hat{U}(t) = e^{-\frac{i}{\hbar} t \hat{H}} = e^{-\frac{i}{\hbar} t \left(\frac{\hat{P}_x^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2\right)}$$

$$\hat{\vec{P}}_S = \hat{\vec{P}} \equiv (\hat{P}_x, \hat{P}_y, \hat{P}_z) \quad ; \quad \frac{\partial \hat{\vec{P}}_S}{\partial t} = 0$$

$$[\hat{U}, \hat{P}_x] \neq 0$$

$$[\hat{U}, \hat{P}_y] = 0 \Rightarrow \hat{P}_{Hy}(t) = \hat{U}^{+}(t)\,\hat{P}_y\,\hat{U}(t) = \hat{P}_y \equiv \hat{P}_{Sy}$$

$$[\hat{U}, \hat{P}_z] = 0 \Rightarrow \hat{P}_{Hz}(t) = \hat{U}^{+}(t)\,\hat{P}_z\,\hat{U}(t) = \hat{P}_z \equiv \hat{P}_{Sz}$$

$[\hat{U}, \hat{p}_x] \neq 0 \quad \Rightarrow$

$$\hat{p}_{Hx}(t) = e^{\frac{i}{\hbar}t\left(\frac{\hat{p}_x^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2\right)}\,\hat{p}_x\,e^{-\frac{i}{\hbar}t\left(\frac{\hat{p}_x^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2\right)}$$

$$\Gamma\; e^{\hat{A}}\hat{B}e^{-\hat{A}} = \sum_{n=0}^{\infty}\frac{1}{n!}[\hat{A},[\hat{A},[\hat{A},\ldots[\hat{A},\hat{B}]\ldots]]] =$$

$$= \hat{B} + [\hat{A},\hat{B}] + \frac{1}{2!}[\hat{A},[\hat{A},\hat{B}]] + \frac{1}{3!}[\hat{A},[\hat{A},[\hat{A},\hat{B}]]] + \ldots$$

$$= \hat{p}_x + \left[\frac{i}{\hbar}t\left(\frac{\hat{p}_x^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2\right), \hat{p}_x\right] +$$

$$\frac{1}{2!}\left[\frac{i}{\hbar}t\left(\frac{\hat{p}_x^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2\right), \left[\frac{i}{\hbar}t\left(\frac{\hat{p}_x^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2\right), \hat{p}_x\right]\right] +$$

$$\left[\frac{i}{\hbar}t\left(\frac{\hat{p}_x^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2\right), \left[\frac{i}{\hbar}t\left(\frac{\hat{p}_x^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2\right), \left[\frac{i}{\hbar}t\left(\frac{\hat{p}_x^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2\right), \hat{p}_x\right]\right]\right]$$

$+ \ldots$

I Komutator $\frac{i}{\hbar}t[\hat{H}, \hat{p}_x] =$

$$\frac{i}{\hbar}t\left[\frac{\hat{p}_x^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2, \hat{p}_x\right] = \frac{i}{\hbar}t\left(\frac{1}{2m}[\hat{p}_x^2, \hat{p}_x] +\right.$$

$$\left. + \frac{1}{2}m\omega^2[\hat{x}^2, \hat{p}_x]\right) = \frac{i}{\hbar}t\,\frac{1}{2}m\omega^2\left([\hat{x},\hat{p}_x]\hat{x} + \hat{x}[\hat{x},\hat{p}_x]\right)$$

$$= \frac{i}{\hbar}t\,\frac{1}{2}m\omega^2\,2i\hbar\hat{x} = -t\,m\omega^2\hat{x}$$

V. komutator

$$\left[\frac{i}{\hbar} t \left(\frac{\hat{p}_x^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2\right), \omega^4 t^4 \hat{p}_x \right] =$$

$$\frac{i}{\hbar} t \frac{1}{2} m\omega^2 \left[\hat{x}^2, \hat{p}_x\right] \omega^4 t^4 = \frac{it}{\hbar} \frac{1}{2} m\omega^2 \, 2i\hbar \hat{x} \, \omega^4 t^4$$

$$= - m\omega^6 t^5 \hat{x}$$

$$\hat{p}_H(t) = \hat{p}_x - t m\omega^2 \hat{x} - \frac{1}{2} \omega^2 t^2 \hat{p}_x + \frac{1}{3!} m\omega^4 t^3 \hat{x} +$$

$$\frac{1}{4!} \omega^4 t^4 \hat{p}_x - \frac{1}{5!} m\omega^6 t^5 \hat{x} + \dots$$

$$= \hat{p}_x \left(1 - \frac{1}{2!} \omega^2 t^2 + \frac{1}{4!} \omega^4 t^4 \right) -$$

$$- \hat{x} \left(t m\omega^2 - \frac{1}{3!} m\omega^4 t^3 + \frac{1}{5!} m\omega^6 t^5 + \dots \right)$$

$$= \hat{p}_x \cos\omega t - m\omega \hat{x} \sin\omega t$$

20. Rešiti Hajzenbergove jne kretanja za opservable položaja i impulsa ($\hat{\vec{r}}$ i $\hat{\vec{p}}$) za

a) slobodnu česticu

b) LHO

$$i\hbar \frac{d\hat{A}_H(t)}{dt} = [\hat{A}_H(t), \hat{H}_H(t)] + i\hbar \hat{U}^+(t) \frac{\partial \hat{A}_S}{\partial t} \hat{U}(t)$$

$$\frac{\partial \hat{\vec{r}}_S}{\partial t} = \frac{\partial \vec{P}_S}{\partial t} = 0 \iff \frac{\partial \hat{A}_S}{\partial t} = 0$$

$$i\hbar \frac{d\hat{A}_H(t)}{dt} = \hat{U}^+(t) [\hat{A}_S, \hat{H}_S] \hat{U}(t)$$

a) $\hat{U}(t) = e^{-\frac{i}{\hbar} t \hat{H}_S} = e^{-\frac{i}{\hbar} t \frac{\hat{\vec{P}}_S^2}{2m}}$

$$i\hbar \frac{d\hat{\vec{r}}_H(t)}{dt} = \hat{U}^+(t) [\hat{\vec{r}}, \frac{\hat{\vec{P}}_S^2}{2m}] \hat{U}(t)$$

$$= \frac{1}{2m} \hat{U}^+(t) [\hat{\vec{r}}, \hat{\vec{P}}_S^2] \hat{U}(t) \qquad \left[[\hat{\vec{r}}, \hat{A}(\hat{\vec{r}}, \hat{\vec{p}})] = i\hbar \frac{\partial \hat{A}}{\partial \vec{p}} \right]$$

$$= \frac{1}{2m} 2i\hbar \hat{U}^+(t) \hat{\vec{P}}_S \hat{U}(t)$$

$$= \frac{i\hbar}{m} \hat{\vec{P}}_S$$

$$i\hbar \frac{d\hat{\vec{r}}_H(t)}{dt} = \frac{i\hbar}{m} \hat{\vec{P}}_S \rightarrow \frac{d\hat{\vec{r}}_H(t)}{dt} = \frac{\hat{\vec{P}}_S}{m}$$

$$i\hbar \frac{d\hat{\vec{P}}_H(t)}{dt} = \hat{U}^+(t) [\hat{\vec{P}}_S, \frac{\hat{\vec{P}}_S^2}{2m}] \hat{U}(t) = 0$$

$$\frac{d\hat{\vec{P}}_H(t)}{dt} = 0 \qquad \frac{d\hat{P}_{iH}(t)}{dt} = 0 \qquad (*)$$

$$\frac{d\hat{\vec{r}}_H(t)}{dt} = \frac{\hat{\vec{P}}_S}{m} \qquad \frac{d\hat{r}_{iH}(t)}{dt} = \frac{\hat{P}_{iS}}{m} \qquad (**)$$

$$(*) \Rightarrow \quad \hat{P}_{iH}(t) = \text{const} \Rightarrow \boxed{\hat{P}_{iH}(t=0) = \hat{P}_{iS}}$$

$$(**) \Rightarrow \quad \hat{r}_{iH}(t) = \frac{\hat{P}_{iS}}{m} t + C \Rightarrow \hat{r}_{iH}(0) = C = \hat{r}_{iS}$$

$$\boxed{\hat{r}_{iH}(t) = \frac{\hat{P}_{iS}}{m} t + \hat{r}_{iS}}$$

b) $$\hat{U}(t) = e^{-\frac{i}{\hbar} t \hat{H}_S} = e^{-\frac{i}{\hbar} t \left(\frac{\hat{P}_S^2}{2m} + \frac{1}{2} m\omega^2 \hat{X}_S^2\right)}$$

$$i\hbar \frac{d\hat{X}_H(t)}{dt} = \hat{U}^+ [\hat{X}_S, \hat{H}_S] \hat{U}$$

$$i\hbar \frac{d\hat{P}_H(t)}{dt} = \hat{U}^+ [\hat{P}_S, \hat{H}_S] \hat{U}$$

$$[\hat{X}_S, \hat{H}_S] = [\hat{X}_S, \frac{\hat{P}_S^2}{2m} + \frac{1}{2} m\omega^2 \hat{X}_S^2] =$$

$$= \frac{1}{2m} [\hat{X}_S, \hat{P}_S^2] = \frac{2i\hbar \hat{P}_S}{2m} = \frac{i\hbar}{m} \hat{P}_S$$

$$\left\{ \begin{array}{l} [\hat{\vec{r}}, \hat{A}(\hat{\vec{r}}, \hat{\vec{P}})] = i\hbar \frac{\partial \hat{A}}{\partial \vec{P}} \\ [\hat{\vec{P}}, \hat{A}(\vec{r}, \vec{P})] = -i\hbar \frac{\partial \hat{A}}{\partial \vec{r}} \end{array} \right.$$

$$[\hat{P}_S, \hat{H}_S] = [\hat{P}_S, \frac{\hat{P}_S^2}{2m} + \frac{1}{2} m\omega^2 \hat{X}_S^2]$$

$$= \frac{1}{2} m\omega^2 [\hat{P}_S, \hat{X}_S^2] = (-2i\hbar \hat{X}_S) \frac{m\omega^2}{\cancel{2}}$$

$$= -i\hbar m\omega^2 \hat{X}_S$$

$$i\hbar \frac{d\hat{X}_H(t)}{dt} = \hat{U}^+ \frac{i\hbar}{m} \hat{P}_S \hat{U} = \frac{i\hbar}{m} \hat{P}_H(t)$$

$$i\hbar \frac{d\hat{P}_H(t)}{dt} = \hat{U} \, -i\hbar m\omega^2 \hat{X}_S \, \hat{U} = -i\hbar m\omega^2 \hat{X}_H(t)$$

$$\begin{cases} \dfrac{d\hat{X}_H(t)}{dt} = \dfrac{\hat{P}_H(t)}{m} & (1) \\ \dfrac{d\hat{P}_H(t)}{dt} = -m\omega^2 \hat{X}_H(t) & (2) \end{cases}$$

$$\frac{d}{dt}(1) \leftarrow (2) \quad \Rightarrow$$

$$\frac{d^2 \hat{X}_H(t)}{dt^2} = \frac{d\hat{P}_H(t)}{m\,dt} = -\frac{m\omega^2}{m} \hat{X}_H(t) = -\omega^2 \hat{X}_H(t)$$

$$\frac{d^2\hat{X}_H(t)}{dt} + \omega^2 \hat{X}_H(t) = 0$$

$$\hat{X}_H(t) = \hat{C}_1 \cos\omega t + \hat{C}_2 \sin\omega t \qquad (3)$$

$$\frac{d}{dt}(2) \leftarrow (1)$$

$$\frac{d^2\hat{P}_H(t)}{dt^2} = -m\omega^2 \frac{d\hat{X}_H(t)}{dt} = -m\omega^2 \frac{\hat{P}_H(t)}{m} = -\omega^2 \hat{P}_H(t)$$

$$\frac{d^2\hat{P}_H(t)}{dt^2} + \omega^2 \hat{P}_H(t) = 0$$

$$\hat{P}_H(t) = \hat{D}_1 \cos\omega t + \hat{D}_2 \sin\omega t \qquad (4)$$

A) $\hat{X}_H(t=0) = \hat{X}_S$

$\hat{P}_H(t=0) = \hat{P}_S$

B) $\left.\dfrac{d\hat{X}_H(t)}{dt}\right|_{t=0}$, $\left.\dfrac{d\hat{P}_H(t)}{dt}\right|_{t=0}$ } почетни услови ↓ честице (а не таласи где су потребни гранични услови)

A)

$$\hat{X}_H(0) = \hat{C}_1 \Rightarrow \hat{C}_1 = \hat{X}_S$$

$$\hat{P}_H(0) = \hat{D}_1 \Rightarrow \hat{D}_1 = \hat{P}_S$$

B)

$$(3) \Rightarrow \frac{d\hat{X}_H(t)}{dt} = -\omega \hat{C}_1 \sin\omega t + \omega \hat{C}_2 \cos\omega t$$

$$\left.\frac{d\hat{X}_H(t)}{dt}\right|_{t=0} = \omega \hat{C}_2$$

$$(1) \quad \left.\frac{d\hat{X}_H(t)}{dt}\right|_{t=0} = \frac{\hat{P}_H(0)}{m} = \frac{\hat{P}_S}{m}$$

$$\Rightarrow \hat{C}_2 = \frac{\hat{P}_S}{m\omega}$$

Rešenje za $\hat{X}_H(t)$

$$\boxed{\hat{X}_H(t) = \hat{X}_S \cos\omega t + \frac{\hat{P}_S}{m\omega}\sin\omega t}$$

$$(4) \Rightarrow \frac{d\hat{P}_H(t)}{dt} = -\omega \hat{D}_1 \sin\omega t + \omega \hat{D}_2 \cos\omega t \Rightarrow \left.\frac{d\hat{P}_H(t)}{dt}\right|_{t=0} = \omega \hat{D}_2$$

$$(2) \Rightarrow \left.\frac{d\hat{P}_H(t)}{dt}\right|_{t=0} = -m\omega^2 \hat{X}_H(0) = -m\omega^2 \hat{X}_S$$

$$\Downarrow$$

$$\hat{D}_2 = -m\omega \hat{X}_S$$

$$\boxed{\hat{P}_H(t) = \hat{P}_S \cos\omega t - m\omega \hat{X}_S \sin\omega t}$$

Uporediti sa 8b

11. Izračunati komutatore

a) $[\hat{X}_H(t_1), \hat{X}_H(t_2)]$

b) $[\hat{P}_H(t_1), \hat{P}_H(t_2)]$

c) $[\hat{X}_H(t_1), \hat{P}_H(t_2)]$

d) $[\hat{H}_H(t_1), \hat{X}_H(t_2)]$

e) $[\hat{P}_H(t_1), \hat{H}_H(t_2)]$ za slobodnu česticu (1D)

Prvo treba računati

$$\hat{X}_H(t) = \hat{U}^+(t)\,\hat{X}\,\hat{U}(t) \quad i \quad \hat{P}_{xH}(t) = \hat{U}^+(t)\,\hat{P}_x\,\hat{U}(t)$$

Slobodna čestica $\quad \hat{U}(t) = e^{-\frac{i}{\hbar}t\hat{H}} = e^{-\frac{i}{\hbar}t\frac{\hat{P}_x^2}{2m}}$

$$\hat{X}_H(t) = e^{\frac{i}{\hbar}t\frac{\hat{P}_x^2}{2m}}\,\hat{X}\,e^{-\frac{i}{\hbar}t\frac{\hat{P}_x^2}{2m}} = \hat{X} + \frac{i}{\hbar}t\left[\frac{\hat{P}_x^2}{2m}, \hat{X}\right] +$$

$$\frac{-1t^2}{2!\hbar^2}\left[\frac{\hat{P}_x^2}{2m}, \left[\frac{\hat{P}_x^2}{2m}, \hat{X}\right]\right] + \dots$$

$$= \hat{X} + \frac{i}{\cancel{\hbar}}t\left(\frac{-2i\cancel{\hbar}\hat{P}_x}{2m}\right) = \hat{X} + \frac{t}{m}\hat{P}_x$$

$$\hat{P}_{xH}(t) = \hat{P}_{xS} = \hat{P}_x$$

a) $[\hat{X}_H(t_1), \hat{X}_H(t_2)] = [\hat{x} + \frac{t_1}{m}\hat{p}_x, \hat{x} + \frac{t_2}{m}\hat{p}_x] =$

$= \dots = \frac{i\hbar}{m}(t_2 - t_1)$

b) $[\hat{p}_H(t_1), \hat{p}_H(t_2)] = 0$

c) $[\hat{X}_H(t_1), \hat{p}_H(t_2)] = [\hat{x} + \frac{t_1}{m}\hat{p}_x, \hat{p}_x] = i\hbar$

d)

$\hat{H}_H(t) = \hat{U}^+(t)\,\hat{H}\,\hat{U}(t) = \hat{U}^+(t)\,\frac{\hat{p}_x^2}{2m}\,\hat{U}(t) = \hat{H}$

$[\hat{H}_H(t_1), \hat{X}_H(t_2)] = [\frac{\hat{p}_x^2}{2m}, \hat{x} + \frac{t}{m}\hat{p}_x] = \frac{1}{2m}[\hat{p}_x^2, \hat{x}]$

$= \frac{1}{2m}(-2i\hbar\hat{p}_x) = -\frac{i\hbar}{m}\hat{p}_x$

e) $[\hat{p}_H(t_1), \hat{H}_H(t_2)] = [\hat{p}_x, \hat{H}] = 0$

12. Isto kao prethodni zadatak, ali za LHO

$$\hat{H} = \frac{\hat{p}_x^2}{2m} + \frac{1}{2} m \omega^2 \hat{x}^2$$

$$\hat{H}_H = \hat{H}_S$$

Prethodno treba izračunati

$$\hat{x}_H(t) = \hat{U}^+(t)\, \hat{x}\, \hat{U}(t) = \dots = \hat{x} \cos\omega t + \frac{\hat{p}_x}{m\omega} \sin\omega t$$

$$\hat{p}_{xH}(t) = \hat{U}^+(t)\, \hat{p}_x\, \hat{U}(t) = \dots = \hat{p}_x \cos\omega t - m\omega\, \hat{x} \sin\omega t$$

a) $$[\hat{x}_H(t_1), \hat{x}_H(t_2)] = [\hat{x}\cos\omega t_1 + \frac{\hat{p}_x}{m\omega}\sin\omega t_1, \hat{x}\cos\omega t_2 + \frac{\hat{p}_x}{m\omega}\sin\omega t_2] =$$

$$\boxed{[\hat{A}+\hat{B}, \hat{C}+\hat{D}] = [\hat{A},\hat{C}] + [\hat{A},\hat{D}] + [\hat{B},\hat{C}] + [\hat{B},\hat{D}]}$$

$$\cos\omega t_1 \cos\omega t_2 [\hat{x},\hat{x}] + \frac{[\hat{x},\hat{p}_x]}{m\omega} \cos\omega t_1 \sin\omega t_2 + \frac{\sin\omega t_1 \cos\omega t_2}{m\omega} [\hat{p}_x,\hat{x}]$$

$$+ \frac{\sin\omega t_1 \sin\omega t_2}{m^2\omega^2} [\hat{p}_x,\hat{p}_x] = \frac{i\hbar}{m\omega} \left(\cos\omega t_1 \sin\omega t_2 - \sin\omega t_1 \cos\omega t_2 \right)$$

$$= \frac{i\hbar}{m\omega} \sin\omega(t_2 - t_1)$$

Ostalo za domaći

13 Izračunati sledeće komutatore za slobodnu česticu i LHO

a) $[\hat{X}_H(t_1), \hat{P}_{x_H}(t_2)]$

b) $[\hat{H}_H(t_1), \hat{X}_H(t_2)]$

c) $[\hat{P}_{x_H}(t_1), \hat{H}_H(t_2)]$

stavljajući da je $t_1 = t_2 = t \ll 1$.

Jednačine kretanja za slobodnu česticu

$$\hat{X}_H(t) = \frac{\hat{P}_x}{m} t + \hat{X} \qquad \hat{H}_H = \hat{H}_S = \frac{\hat{P}_x^2}{2m}$$

$$\hat{P}_{x_H}(t) = \hat{P}_x$$

a) $[\hat{X}_H(t), \hat{P}_{x_H}(t)] = [\frac{\hat{P}_x}{m} t + \hat{X}, \hat{P}_x] = i\hbar$

b) $[\hat{H}_H(t), \hat{X}_H(t)] = [\frac{\hat{P}_x^2}{2m}, \frac{\hat{P}_x}{m} t + \hat{X}] =$

$$= \frac{1}{2m}[\hat{P}_x^2, \hat{X}] = -\frac{1}{2m}[\hat{X}, \hat{P}_x^2] = -\frac{1}{2m} i\hbar\, 2\hat{P}_x$$
$$= -\frac{i\hbar}{m}\hat{P}_x$$

c) $[\hat{P}_{x_H}(t), \hat{H}_H(t)] = [\hat{P}_x, \frac{\hat{P}_x^2}{2m}] = 0$

Jednačine kretanja za LHO

$$\hat{X}_H(t) = \hat{X}_S \cos\omega t + \frac{\hat{P}_S}{m\omega}\sin\omega t$$

$$\hat{P}_H(t) = \hat{P}_S \cos\omega t - m\omega \hat{x}_S \sin\omega t$$

$$\hat{H}_H = \hat{H}_S = \frac{\hat{P}_x^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2$$

$$\boxed{\begin{matrix} \cos\omega t \sim 1 \\ \sin\omega t \sim \omega t \end{matrix}}$$

a) $$[\hat{X}_H(t), \hat{P}_{xH}(t)] = [\hat{X}_{S} \cos\omega t + \frac{\hat{P}_x}{m\omega} \sin\omega t,$$

$$\hat{P}_x \cos\omega t - m\omega \hat{x} \sin\omega t] \overset{t\ll 1}{=} [\hat{x} + \frac{\hat{P}_x}{m\omega}\omega t, \hat{P}_x - m\omega^2 t \hat{x}]$$

$$= [\hat{x}, \hat{P}_x] - \omega^2 t^2 (-i\hbar) = i\hbar(1 + \omega^2 t^2)$$

$$\boxed{\text{Domaći}}$$

b) $$[\hat{H}_H(t), \hat{X}_H(t)] = [\frac{\hat{P}_x^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2, \hat{x}\cos\omega t + \frac{\hat{P}_x}{m\omega}\sin\omega t]$$

$$\overset{t\ll 1}{=} [\frac{\hat{P}_x^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2, \hat{x} + \frac{\hat{P}_x t}{m}] = [\frac{\hat{P}_x^2}{2m}, \hat{x}] + [\frac{1}{2} m\omega^2 \hat{x}^2, \frac{\hat{P}_x t}{m}]$$

$$= -\frac{1}{2m}[\hat{x}, \hat{P}_x^2] + \frac{\omega^2 t}{2}[\hat{P}_x, \hat{x}^2] =$$

$$= -\frac{1}{2m}(i\hbar(2\hat{P}_x)) - \frac{\omega^2 t}{2}(-i\hbar(2\hat{x})) =$$

$$= -\frac{i\hbar}{m}\hat{P}_x + \frac{i\omega^2 \hbar t}{2}\hat{x}$$

c) $$[\hat{P}_{xH}(t), \hat{H}_H(t)] = [\hat{P}_x \cos\omega t - m\omega \hat{x} \sin\omega t, \frac{\hat{P}_x^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2]$$

$$\overset{t\ll 1}{=} [\hat{P}_x - m\omega^2 t \hat{x}, \frac{\hat{P}_x^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2]$$

$$= [\hat{P}_x, \frac{1}{2} m\omega^2 \hat{x}^2] - m\omega^2 t [\hat{x}, \frac{\hat{P}_x^2}{2m}] =$$

$$= \frac{1}{2} m\omega^2 [\hat{p}_x, \hat{x}^2] - \frac{\cancel{m}\omega^2 t}{2\cancel{m}} [\hat{x}, \hat{p}_x^2] =$$

$$= \frac{1}{\cancel{2}} m\omega^2 (-i\hbar \cancel{2}\hat{x}) - \frac{\omega^2 t}{\cancel{2}} (i\hbar \cancel{2}\hat{p}_x)$$

$$= -i\hbar m\omega^2 \hat{x} - i\hbar \omega^2 t \hat{p}_x$$

14. Za dato stanje $|\Psi\rangle$ konzervativnog sistema naći verovatnoću da se merenjem energije u trenutku t dobije rezultat E_n iz diskretnog spektra svojstvenih vrednosti.

$$W(\hat{H}, |\Psi\rangle, t, E_n) = ?$$

$$W(\hat{H}, |\Psi\rangle, t, E_n) = \langle \Psi(t) | \hat{P}_n | \Psi(t) \rangle$$

$$|\Psi(t)\rangle = \hat{U}(t) |\Psi(t=0)\rangle$$

$$\langle \Psi(t)| = \langle \Psi(t=0) | \hat{U}^{+}(t)$$

$$W(\dots) = \langle \Psi(t=0) | \hat{U}^{+}(t) \hat{P}_n \hat{U}(t) | \Psi(t=0) \rangle$$

$$\hat{U}(t) = e^{-\frac{i}{\hbar} t \hat{H}} = \sum_{K} e^{-\frac{i}{\hbar} E_K t} \hat{P}_K \; ; \quad \hat{H} = \sum_{K} E_K \hat{P}_K$$

$$[\hat{U}(t), \hat{P}_n] = \sum_{K} e^{-\frac{i}{\hbar} E_K t} [\hat{P}_K, \hat{P}_n] = 0 \Rightarrow$$

$$\hat{U}(t) \hat{P}_n = \hat{P}_n \hat{U}(t) \Rightarrow \hat{P}_n = \hat{U}^{+}(t) \hat{P}_n \hat{U}(t)$$

$$W(\hat{H}, |\Psi\rangle, t, E_n) = \langle \Psi(t=0) | \hat{P}_n | \Psi(t=0) \rangle$$

$$= W(\hat{H}, |\Psi\rangle, t=0, E_n)$$

15. Eksplicitno napisati ŠJ (u koordinatnoj rep.) za naelektrisanu česticu u spoljašnjem (homogenom i vremenski nez.) magnetnom polju definisanim vektorskim potencijalom $\vec{A}$.

$$\vec{B} = \nabla \times \vec{A} \qquad \vec{E} = \frac{\partial \vec{A}}{\partial t} + \vec{\nabla}\phi \; ; \; \vec{\nabla} = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}$$

$$\vec{\underline{P}} = \vec{p} - \frac{e}{c}\vec{A}(\vec{r})$$

$$H = \frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}(\vec{r})\right)^2 + e\phi(\vec{r}) + V(\vec{r})$$

Ovde je magnetno polje homogeno i ne menja se sa vremenom. i nema drugih potencijala

$$H = \frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}(\vec{r})\right)^2$$

Kvantizacija

$$\hat{H} = \frac{1}{2m}\left(\hat{\vec{p}} - \frac{e}{c}\vec{A}(\hat{\vec{r}})\right)^2$$

Šredingerova j-na

$$\hat{H}|\Psi\rangle = E_n|\Psi\rangle$$

U koordinatnoj reprezentaciji

$$\hat{\vec{p}} = -i\hbar\vec{\nabla}$$

ŠJ glasi

$$\frac{1}{2m}\left(-i\hbar\vec{\nabla} - \frac{e}{c}\vec{A}(\vec{r})\right)^2\Psi(\vec{r}) = E_n\Psi(\vec{r})$$

16. Naelektrisana čestica se kreće u homogenom magnetnom polju. Naći operator vektora položaja $\vec{r}_H(t)$ ukoliko je vektorski potencijal dat kao $\vec{A} = (0, B\cdot x, 0)$, gde je B konstanta.

$$\hat{\vec{r}}_H(t) = \hat{U}^+(t)\,\hat{\vec{r}}\,\hat{U}(t)$$

$$\hat{U} = e^{-\frac{i}{\hbar}\hat{H}t}$$

$$\hat{\vec{r}}_H(t) = e^{+\frac{i}{\hbar}t\hat{H}}\,\hat{\vec{r}}\,e^{-\frac{i}{\hbar}t\hat{H}}$$

$$\boxed{e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + [\hat{A},\hat{B}] + \frac{1}{2!}[\hat{A},[\hat{A},\hat{B}]] + \dots}$$

$$\hat{\vec{r}}_H(t) = \vec{r} + \frac{i}{\hbar}t[\hat{H},\hat{\vec{r}}] - \frac{t^2}{2\hbar^2}[\hat{H},[\hat{H},\hat{\vec{r}}]] + \dots$$

$$\hat{H} = \frac{1}{2m}\left(\hat{\vec{p}} - \frac{e}{c}\vec{A}(\hat{\vec{r}})\right)^2$$

$$= \frac{1}{2m}\left(\hat{\vec{p}} - \frac{e}{c}\vec{A}(\hat{\vec{r}})\right)\left(\hat{\vec{p}} - \frac{e}{c}\vec{A}(\hat{\vec{r}})\right)$$

$$= \frac{1}{2m}\left(\hat{\vec{p}}^2 - \frac{e}{c}\hat{\vec{p}}\vec{A}(\hat{\vec{r}}) - \frac{e}{c}\vec{A}(\hat{\vec{r}})\hat{\vec{p}} + \frac{e^2}{c^2}\vec{A}^2(\hat{\vec{r}})\right)$$

$$= \frac{1}{2m}\hat{\vec{p}}^2 - \frac{e}{2mc}\left(\hat{\vec{p}}\cdot\vec{A}(\hat{\vec{r}}) + \vec{A}(\hat{\vec{r}})\cdot\hat{\vec{p}}\right) + \frac{e^2}{2mc^2}\vec{A}^2(\hat{\vec{r}})$$

$$\vec{A} = (0, B\cdot x, 0) \Rightarrow \vec{A}(\hat{x}) = (0, B\hat{x}, 0)$$

$$\hat{\vec{p}}\vec{A} = B\hat{p}_y\hat{x} \Rightarrow \vec{A}\hat{\vec{p}} = B\hat{x}\hat{p}_y$$

$$\vec{A}^2(\hat{\vec{r}}) = A^2(\hat{x}) = B^2\hat{x}^2$$

$$\hat{H} = \frac{\hat{\vec{p}}^2}{2m} - \frac{Be}{2mc}(\hat{p}_y\hat{x} + \hat{x}\hat{p}_y) + \frac{e^2B^2}{2mc^2}\hat{x}^2$$

$$\omega = eB/mc$$

$$\hat{H} = \frac{\hat{\vec{p}}^2}{2m} - \frac{\omega}{2}(\hat{p}_y\hat{x} + \hat{x}\hat{p}_y) + \frac{m\omega^2}{2}\hat{x}^2$$

$$[\hat{H}, \hat{\vec{r}}] = ?$$

$$[\hat{H}, \hat{x}] = \left[\frac{\hat{\vec{p}}^2}{2m} - \frac{\omega}{2}(\hat{p}_y\hat{x} + \hat{x}\hat{p}_y) + \frac{m\omega^2}{2}\hat{x}^2, \hat{x}\right]$$

$$= \left[\frac{\hat{\vec{p}}^2}{2m}, \hat{x}\right] - \frac{\omega}{2}\left([p_y\cancel{\hat{x}}, \hat{x}] + [\hat{x}\hat{p}_y, \hat{x}]\right)$$

$$= \left[\frac{\hat{p}_x^2}{2m}, \hat{x}\right] = -\frac{1}{2m}[\hat{x}, \hat{p}_x^2] = -\frac{1}{2m}2i\hbar\hat{p}_x$$

$$= -\frac{i\hbar}{m}\hat{p}_x$$

$$[\hat{H}, \hat{y}] = -\frac{i\hbar}{m}\hat{p}_y$$

КОРИГОВАТИ
ФАЛИ $i\hbar\omega\hat{x}$ И НАСТАВИТИ ДАЉЕ

$$[\hat{H}, \hat{z}] = -\frac{i\hbar}{m}\hat{p}_z$$

$$\boxed{[\hat{H}, \hat{\vec{r}}] = -\frac{i\hbar}{m}\hat{\vec{p}}}$$

$$[\hat{H},[\hat{H},\hat{X}]] = [\hat{H}, -\frac{i\hbar}{m}\hat{P}_x] = [\frac{\hat{\vec{P}}^2}{2m} - \frac{\omega}{2}(\hat{P}_y\hat{X}+\hat{X}\hat{P}_y) + \frac{m\omega^2}{2}\hat{X}^2, -\frac{i\hbar}{m}\hat{P}_x] =$$

$$= -\frac{i\hbar}{m}\cancel{[\hat{\vec{P}}^2, \hat{P}_x]}^{0} + \frac{i\hbar}{m}\frac{\omega}{2}[\hat{P}_y\hat{X},\hat{P}_x] + \frac{i\hbar}{m}\frac{\omega}{2}[\hat{X}\hat{P}_y,\hat{P}_x] - \frac{i\hbar}{2}\omega^2[\hat{X}^2,\hat{P}_x] =$$

$$= \frac{i\hbar}{m}\frac{\omega}{2}(\cancel{[\hat{P}_y,\hat{P}_x]}^{0}\hat{X} + \hat{P}_y[\hat{X},\hat{P}_x]) + \frac{i\hbar}{m}\frac{\omega}{2}([\hat{X},\hat{P}_x]\hat{P}_y + \hat{X}\cancel{[\hat{P}_y,\hat{P}_x]}) + \frac{i\hbar}{2}\omega^2[\hat{P}_x,\hat{X}^2] =$$

$$= \frac{i\hbar}{m}\frac{\omega}{2}i\hbar\hat{P}_y + \frac{i\hbar}{m}\frac{\omega}{2}i\hbar\hat{P}_y + \frac{i\hbar}{2}\omega^2(-2i\hbar\hat{X}) =$$

$$= \frac{(i\hbar)^2}{m}\omega\hat{P}_y + (i\hbar)^2\omega^2\hat{X}$$

$$= -\frac{\hbar^2}{m}\omega\hat{P}_y + \hbar^2\omega^2\hat{X}$$

$$[\hat{H},[\hat{H},[\hat{H},\hat{X}]]] = [\hat{H}, -\frac{\hbar^2}{m}\omega\hat{P}_y + \hbar^2\omega^2\hat{X}] =$$

$$= -\frac{\hbar^2}{m}[\hat{H},\omega\hat{P}_y] + \hbar^2\omega^2\underbrace{[\hat{H},\hat{X}]}_{\text{Poznato iz prethodnog koraka.}}$$

$$= -\frac{\hbar^2}{m}[\hat{H},\omega\hat{P}_y] + \hbar^2\omega^2\left(-\frac{i\hbar}{m}\right)\hat{P}_x$$

$$= -\frac{\hbar^2\omega}{m}[\hat{H},\hat{P}_y] - \frac{i\hbar^3\omega^2}{m}\hat{P}_x$$

$$= -\frac{\hbar^2\omega}{m}\left[\frac{\hat{\vec{P}}^2}{2m} - \frac{\omega}{2}(\hat{P}_y\hat{X}+\hat{X}\hat{P}_y) + \frac{m\omega^2}{2}\hat{X}^2, \hat{P}_y\right] - i\frac{\hbar^3\omega^2}{m}\hat{P}_x$$

$$= -\frac{\hbar^2\omega}{m}\cancel{\left[\frac{\hat{\vec{P}}^2}{2m}, \hat{P}_y\right]}^{0} + \frac{\hbar^2\omega^2}{2m}[\hat{P}_y\hat{X}, \hat{P}_y] + \frac{\hbar^2\omega^2}{2m}[\hat{X}\hat{P}_y, \hat{P}_y]$$

$$- \frac{\hbar^2\omega^3}{2}\cancel{[\hat{X}^2, \hat{P}_y]}^{0} - \frac{i\hbar^3\omega^2}{m}\hat{P}_x =$$

$$= \frac{\hbar^2\omega^2}{2m}\left(\cancel{[\hat{P}_y, \hat{P}_y]}^{0}\hat{X} + \hat{P}_y\cancel{[\hat{X}, \hat{P}_y]}^{0}\right) + \frac{\hbar^2\omega^2}{2m}\left(\cancel{[\hat{X}, \hat{P}_y]}^{0}\hat{P}_y + \right.$$

$$\left.\hat{X}\cancel{[\hat{P}_y, \hat{P}_y]}^{0}\right) - i\frac{\hbar^3\omega^2}{m}\hat{P}_x = -i\frac{\hbar^3\omega^2}{m}\hat{P}_x$$

⌈ Gore je dobijeno da je $[\hat{H}, \hat{P}_y] = 0$ ⌋

$$[\hat{H}, [\hat{H}, [\hat{H}, [\hat{H}, X]]]] = \left[\hat{H}, -\frac{i\hbar^3\omega^2}{m}\hat{P}_x\right] =$$

$$-\frac{i\hbar^3\omega^2}{m}[\hat{H}, \hat{P}_x] = -\frac{i\hbar^3\omega^2}{m}\left[\frac{\hat{\vec{P}}^2}{2m} - \frac{\omega}{2}(\hat{P}_y\hat{X}+\hat{X}\hat{P}_y) + \frac{m\omega^2}{2}\hat{X}^2, \hat{P}_x\right]$$

$$-\frac{i\hbar^3\omega^2}{m}\left(\cancel{\left[\frac{\vec{P}^2}{2m}, \hat{P}_x\right]}^{0} - \frac{\omega}{2}[\hat{P}_y\hat{X}, \hat{P}_x] - \frac{\omega}{2}[\hat{X}\hat{P}_y, \hat{P}_x] + \right.$$

$$\left. + \frac{m\omega^2}{2}[\hat{X}^2, \hat{P}_x]\right) =$$

$$= \frac{i\hbar^3\omega^3}{2m}\left(\cancel{[\hat{p}_y,\hat{p}_x]}\hat{x} + \hat{p}_y[\hat{x},\hat{p}_x]\right) + \frac{i\hbar^3\omega^3}{2m}\left([\hat{x},\hat{p}_x]\hat{p}_y + \hat{x}\cancel{[\hat{p}_y,\hat{p}_x]}\right)$$

$$-\frac{i\hbar^3\omega^4}{2}[\hat{x}^2,\hat{p}_x] =$$

$$= \frac{i\hbar^3\omega^3}{\cancel{2}m}\cancel{2}i\hbar\hat{p}_y + \frac{i\hbar^3\omega^4}{2}[\hat{p}_x,\hat{x}^2] =$$

$$= \frac{i\hbar^3\omega^3}{\cancel{2}m}i\hbar\hat{p}_y + \frac{i\hbar^3\omega^4}{\cancel{2}}(-\cancel{2}i\hbar\hat{x}) =$$

$$= -\frac{\hbar^4\omega^3}{m}\hat{p}_y + \hbar^4\omega^4\hat{x}$$

$$\hat{x}_H(t) = \hat{x} + \frac{i}{\hbar}t\left(-\frac{i\hbar}{m}\hat{p}_x\right) - \frac{t^2}{2\hbar^2}\left(-\frac{\hbar^2}{m}\omega\hat{p}_y + \hbar^2\omega^2\hat{x}\right)$$

$$-\frac{it^3}{3!\hbar^3}\left(-i\frac{\hbar^3\omega^2}{m}\hat{p}_x\right) + \frac{t^4}{4!\hbar^4}\left(-\frac{\hbar^4\omega^3}{m}\hat{p}_y + \hbar^4\omega^4\hat{x}\right) + \cdots$$

$$= \hat{x} + \frac{t}{m}\hat{p}_x + \frac{t^2}{2m}\omega\hat{p}_y - \frac{t^2\omega^2}{2}\hat{x} - \frac{t^3\omega^2}{3!m}\hat{p}_x -$$

$$-\frac{1}{4!}\frac{t^4\omega^3}{m}\hat{p}_y + \frac{1}{4!}t^4\omega^4\hat{x} + \cdots$$

$$= \hat{x}\left(1 - \frac{t^2\omega^2}{2!} + \frac{t^4\omega^4}{4!} + \cdots\right) + \frac{\hat{p}_x}{m\omega}\left(t\omega - \frac{t^3\omega^3}{3!} + \cdots\right)$$

$$+ \frac{\hat{p}_y}{m\omega}\left(\frac{t^2\omega^2}{2!} - \frac{t^4\omega^4}{4!} + \cdots\right) =$$

$$\left[\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots,\ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right]$$

$$= \hat{X}\cos\omega t + \frac{\hat{p}_x}{m\omega}\sin\omega t + \frac{\hat{p}_y}{m\omega}\left(1 - 1 + \frac{t^2\omega^2}{2!} - \frac{t^4\omega^4}{4!} + \dots\right)$$

$$= \hat{X}\cos\omega t + \frac{\hat{p}_x}{m\omega}\sin\omega t + \frac{\hat{p}_y}{m\omega}\left(1 - \left(1 - \frac{t^2\omega^2}{2!} + \frac{t^4\omega^4}{4!} - \dots\right)\right)$$

$$= \hat{X}\cos\omega t + \frac{\hat{p}_x}{m\omega}\sin\omega t + \frac{\hat{p}_y}{m\omega}\underline{(1 - \cos\omega t)}$$

$$[\hat{H}, \hat{y}] = -\frac{i\hbar}{m}\hat{p}_y$$

$$[\hat{H}, [\hat{H}, \hat{y}]] = -\frac{i\hbar}{m}\cancel{[\hat{H}, \hat{p}_y]}^{0} = 0$$

v. prethodne korake

$$\hat{y}_H(t) = \hat{y} + \frac{i}{\hbar}t\left(-\frac{i\hbar}{m}\hat{p}_y\right) = \hat{y} + \frac{t}{m}\hat{p}_y$$

Za domaći

$$\hat{z}_H(t) = \hat{z} + \frac{t}{m}\hat{p}_z$$